

## Exercise Sheet 2

**Exercise 1.** (for credit, due on 28 September) Consider the rational function  $f(z) = \frac{(z+1)^2}{z^4}$ .

- (1) (1 point) Show that  $f$  extends to a holomorphic map  $f : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ .
- (2) (4 points) Compute the ramification points (including multiplicities) and branch values of  $f$ .

**Exercise 2.** Let  $f : X \rightarrow Y$  be a non-constant holomorphic map between connected Riemann surfaces.

- (1) Show that the set of ramification points of  $f$  is discrete in  $X$ .
- (2) Assume that  $f$  is also proper (i.e. the inverse image of a compact set is compact). Show that the set of branching points is discrete.
- (3) Assume that  $X = Y$  and  $f \neq \text{id}_X$ . Show that the set of fixed points  $\{p \in X : f(p) = p\}$  is discrete.
- (4) Give an example where the set of ramification points is infinite and the set of branching points is not discrete. **Hint:** Trigonometric functions.

**Exercise 3.** Show that a bijective holomorphic map between two Riemann surfaces is a biholomorphism.

**Exercise 4** (Schwarz lemma). Let  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  be the unit disk and suppose  $f : \mathbb{D} \rightarrow \mathbb{D}$  is holomorphic with  $f(0) = 0$ . Show that  $|f(z)| \leq |z|$  for all  $z \in \mathbb{D}$  and  $|f'(0)| \leq 1$ . Show that if moreover  $|f(z_0)| = |z_0|$  for some non-zero  $z_0 \in \mathbb{D}$  or  $|f'(0)| = 1$ , then  $f(z) = az$  for some  $a \in \mathbb{C}$  with  $|a| = 1$ .

**Exercise 5.** For a Riemann surface  $X$ , the automorphism group  $\text{Aut}(X)$  is the group of biholomorphisms  $f : X \rightarrow X$ . Denote by  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  the unit disk and by  $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$  the upper half-plane. Show that:

$$\begin{aligned} \text{Aut}(\mathbb{CP}^1) &= \left\{ z \mapsto \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{C} \text{ with } ad - bc \neq 0 \right\}; \\ \text{Aut}(\mathbb{C}) &= \{z \mapsto az + b : a, b \in \mathbb{C} \text{ with } a \neq 0\}; \\ \text{Aut}(\mathbb{D}) &= \left\{ z \mapsto \frac{az+b}{\bar{b}z+\bar{a}} : a, b \in \mathbb{C} \text{ with } |a|^2 - |b|^2 = 1 \right\}; \\ \text{Aut}(\mathbb{H}) &= \left\{ z \mapsto \frac{az+b}{cz+d} : a, b, c, d \in \mathbb{R} \text{ with } ad - bc > 0 \right\}. \end{aligned}$$

**Exercise 6.** (challenging)

What is the automorphism group of the three-punctured Riemann sphere  $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$ ?